Lattice Hydrodynamic Model for Traffic Flow: A Review

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Abstract: This present study examines the lattice hydrodynamic model (LHM) as applied to traffic flow theory, focusing on its development, applications, and contributions to understanding vehicular movement on road networks. The LHM provides a framework for modeling traffic dynamics by discretizing both space and time on a grid, representing vehicles as particles in a lattice structure. This approach allows for the simulation of various traffic phenomena, such as congestion, flow breakdown, and shockwave formation. The review traces the evolution of the model from its initial applications to its refinement through various extensions, including the incorporation of driver behavior, road characteristics, and heterogeneous traffic conditions. Key methodologies, computational techniques and challenges in applying the LHM to real-world traffic scenarios are discussed, along with comparisons to other traffic flow models. By synthesizing recent research and advancements, this review highlights the potential of the lattice hydrodynamic model in providing valuable insights for traffic management and optimization strategies while identifying avenues for future research to enhance its accuracy and practical implementation.

Keywords: Lattice Hydrodynamic Model

INTRODUCTION

To develop a new lattice hydrodynamic model, it is essential to study the existing strategies, theories, methodologies and technologies in the field. Therefore, this chapter reviews previous studies on lattice hydrodynamic models (LHM). Here we discuss the process of designing a new LHM model based on existing approaches identifying the research gaps in prior work. Lattice hydrodynamic models treat traffic flow as a continuous stream similar to fluid dynamics rather than focusing on individual vehicles. The interactions among vehicles are governed by mathematical rules that dictate movement, acceleration and deceleration based on traffic density and flow conditions. While lattice models are effective in simulating large-scale traffic patterns they often struggle to accurately capture the complexities of urban driving such as managing intersections handling lane changes and addressing congestion at bottlenecks. Consequently there is a continuous effort to improve these models to make them more capable of representing real-world traffic behaviors in urban and city settings. Since the seminal work of Lighthill, Whitham and Richards (LWR model) on kinematic waves in vehicular traffic flow Payne [1] introduced a high-order continuum traffic flow model including a dynamic equation, which was derived from the car-following theory: Firstly Lighthill, Whitham and Richards (LWR model) derived an equation in vehicular traffic flow after that Payne derived a high-order continuum traffic flow model including a dynamic equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial(v)}{\partial x} = \frac{v_e(\rho) - v}{\tau} - \frac{\mu}{\rho t}\frac{\partial(\rho v)}{\partial x} \tag{2}$$

ρ stands for traffic density, v denotes space mean speed, τ represents relaxation time, $v(\rho)$ signifies equilibrium speed and μ represents the anticipation coefficient $\mu = -0.5, \frac{\partial(v_e(\rho))}{\partial \rho}$. The former term aids in relaxing to equilibrium while the latter mirrors driver reaction to leading cars Nagatani introduced [1] a simplified hydrodynamic model for continuum analysis as following:

$$\partial_t \rho + \rho_0\, \partial_x(\rho v) = 0 \tag{3}$$

$$\partial_t(\rho v) = a\rho_0 V(\rho(x+\delta)) - a\rho v \tag{4}$$

Where $a = \frac{1}{\tau}$, ρ_0, $\rho(x+\delta)$ and $\delta = \frac{1}{\rho_0}$ are the sensitivity of a driver, average density and local density at position $(x+\delta)$ at time t and the average headway $\{h(x,t)\}$. Where $\rho(x+\delta) = \frac{1}{h(x,t)}$

Bando [2] derived a velocity model on the behalf car following model.

$$\frac{d^2 x_j(t)}{dt^2} = a[\widetilde{V}(\Delta\left(x_j(t)\right) - \frac{d(x_j(t))}{dt}] \tag{5}$$

By the approach that a driver can adjust the car velocity by observed the gap between the cars, which corresponds to the inverse of the local density $h(x,t)$. Equation (4) comes from to the Equation (2). Eq.(4) is modified with $\tilde{x} = x/\delta$ where x is dimensionless space and $\tilde{x}$ is indicated as x here after. We know that Lattice hydrodynamic model has three types. Model A is similar to Bando Model [2].

$$\partial_t \rho_j + \rho_0\, \partial_t\left(\rho_j v_j - \rho_{j-1} v_{j-1}\right) = 0 \tag{6}$$

$$\partial_t \rho_j v_j = a\rho_0 V(\rho_{j+1}) - a\rho_j v_j \tag{7}$$

LHM started in 1998 by Nagatani [3], after that this model was used by different researchers to develop new models which enhance traffic flow. Initial models were not so effective but later on the models were made more effective. Enhancement id model was not so significant till 2008. But, work of Hong Xia-Ge and Rong-Jun Cheng initiated effectiveness of LHM.

Review

In 2008, a model on t**he "backward looking" effect in the lattice hydrodynamic model** is derived by the Hong Xia-Ge and Rong-Jun Cheng[4]. In this paper, the novel lattice hydrodynamic model is presented by incorporating the "backward looking" effect. The stability condition for the the model is obtained using the linear stability theory. The result shows that considering one following site in vehicle motion leads to the stabilization of the system compared with the original lattice hydrodynamic model and the cooperative driving lattice hydrodynamic model. The Korteweg–de Vries (KdV, for short) equation near the neutral stability line is derived by using the reductive perturbation method to show the traffic jam which is proved to be described by KdV soliton solution obtained from the KdV equation. The simulation result is consistent with the nonlinear analysis. an extended LH model with just one following vehicle is presented. The linear stability theory is given and the neutral stability condition is obtained. The linear result shows that this model could stabilize the traffic flow perfectly, which is consistent with the backward looking optimal velocity model (for short, BL–OV model). Moreover, we make the nonlinear analysis near the neutral stability line, so the KdV equation and its corresponding soliton solution are obtained. Numerical simulation is carried out to validate the nonlinear result.

In 2009, a research has been done upon the "Lattice hydrodynamic model with bidirectional pedestrian flow" [5]. In this research the two-dimensional lattice hydrodynamic model of traffic is extended to the two-dimensional bidirectional pedestrian flow via taking four types of pedestrians into account. The stability condition and the mKdV equation to describe the density wave of pedestrian congestion are obtained by linear stability and nonlinear analysis, respectively. In addition, there exist three phase transitions among the freely moving phase, the coexisting phase and the uniformly congested phase in the phase diagram. It can also be found that the critical point a_c refers to not only the fraction c_1 of the eastbound and westbound pedestrians, but also the fraction c_2 of the northbound and southbound pedestrians. However, the critical point a_c could not appear in the phase diagram and congested crowd at any time when two fractions are equal to same value of 0.5 ($c_1 = c_2 = 0.5$). Furthermore, numerical simulation is carried out to examine the performance of such a model and the results show coincidence with the theory analysis results. Whether there exist the kink–antikink density waves in pedestrian traffic. The extension of the two-dimensional lattice hydrodynamic model to the two-dimensional bidirectional pedestrian flow is taken into account by introducing the four types of pedestrians. the two-dimensional bidirectional pedestrian flow is proposed. And the characteristics of the two-dimensional pedestrian traffic model are analyzed by using the linear stability theory, nonlinear analysis and verifying by computer simulation respectively.

In 2010, a research has been done upon the "Flow difference effect in the lattice hydrodynamic model" [6]. In this research, the concept of "vehicles move with flow difference" suggests a scenario where vehicles adjust their movements based on the flow differences within a traffic system. This concept can have several uses and future scopes implementing systems where vehicles adjust their speeds and routes based on flow differences can greatly improve traffic management and overall efficiency. By dynamically adapting to changes in traffic flow, vehicles can help alleviate congestion, reduce travel times, and optimize the utilization of road networks. Integrating this concept into smart transportation systems can lead to more responsive and adaptive traffic control mechanisms. Vehicles equipped with advanced sensors and communication technologies can exchange real-time traffic data and coordinate their movements to maintain smooth traffic flow and minimize delays. By synchronizing their movements based on flow differences, vehicles can potentially reduce the likelihood of accidents and collisions. Cooperative adaptive cruise control (CACC) systems, for example, can enable vehicles to maintain safe distances and adjust speeds in unison, enhancing overall road safety. More efficient traffic flow resulting from vehicles moving with flow differences can lead to reduced fuel consumption and emissions. By minimizing stop-and-go traffic patterns and optimizing driving speeds, this concept can contribute to environmental sustainability and air quality improvement. Incorporating the idea of vehicles moving with flow differences into urban planning and infrastructure design can help optimize the layout of roads, intersections, and traffic signal systems. By understanding how traffic flows interact with the built environment, planners can design more efficient and resilient transportation networks. Autonomous vehicles can leverage the concept of moving with flow differences to navigate complex traffic scenarios more effectively. By analyzing real-time traffic data and adjusting their behaviors accordingly, self-driving cars can seamlessly integrate into mixed traffic environments and contribute to smoother overall traffic flow. There is significant scope for further research and development in this area, including refining algorithms for vehicle coordination, studying the impact of different traffic scenarios and road conditions, and exploring the integration of emerging technologies such as connected and automated vehicles. Advancements in this field could lead to innovative solutions for improving mobility and transportation efficiency. Implementing policies and regulations that incentivize or mandate the adoption of technologies enabling vehicles to move with flow differences can have profound implications for transportation systems and urban development. Governments and transportation authorities may need to consider issues related to privacy, data security, infrastructure investment, and public acceptance as they promote the adoption of these concepts.

In 2011, a research has been done upon the "Non-lane-based lattice hydrodynamic model of traffic flow considering the lateral effects of the lane width [7]. In this research a novel approach to traffic modeling has emerged, introducing a non-lane-based lattice model that incorporates the lateral separation effects stemming from lane width. Through rigorous analysis utilizing linear stability theory, the model's stability conditions are derived, shedding light on how traffic flow behaves under varying conditions. By extending this analysis to nonlinear dynamics, a modified Korteweg-de Vries (KdV) equation emerges, providing insights into the phase transitions of traffic flow and the evolution of congestion. Numerical simulations underscore the significance of integrating lane width effects into lattice models. Notably, the simulations reveal that this integration stabilizes traffic flow and mitigates congestion, offering a promising solution to address traffic jams. These findings highlight the pivotal role of lateral separation effects in lattice modeling, influencing car-following behaviors under diverse road conditions. In this communication, an extended non-lane-based lattice model is introduced, specifically considering the lateral effects of lane width. Through a comprehensive analysis encompassing both linear stability and nonlinear dynamics, the study elucidates the efficacy of lateral separation in stabilizing traffic flows within the lattice framework. Simulation outcomes corroborate theoretical analyses, affirming the validity and relevance of the proposed considerations.

In 2011, a research has been done upon the "A new lattice model of traffic flow with the consideration of the driver's forecast effects" [8]. This research introduced a novel lattice model that incorporates the influence of Driver's Forecast Effects (DFE). Through rigorous analysis employing linear stability theory, they established the linear stability condition of this extended model, revealing its capacity to enhance traffic flow stability by considering DFE. Nonlinear analysis further unveils the dynamics of traffic jams through the derivation of a modified Korteweg-de Vries (KdV) equation near critical points. Numerical simulations affirm the model's efficacy in improving traffic flow stability, particularly by adjusting the driver's forecast intensity parameter in response to Intelligent Transportation Systems (ITS) information. Their findings underscore the pivotal role of DFE in forecasting future traffic situations and its potential to bolster traffic flow stability, especially with increasing parameter values. Analytical and numerical investigations jointly demonstrate the beneficial impact of DFE on traffic flow stability, with stability increasing alongside the parameter 'p'. Furthermore, our model effectively captures the evolution and propagation of minor disturbances, while numerical tests underscore that comprehensive consideration of DFE can entirely mitigate unstable traffic flow dynamics.

In 2012, a research has been done upon the "**The stabilization effect of the density difference in the modified lattice hydrodynamic model of traffic flow** [9]. In this research a refined lattice hydrodynamic model for traffic flow is introduced, incorporating density discrepancies between leading and following lattice points. Through linear stability analysis, the model's stability conditions are elucidated, revealing a stabilizing effect induced by considering density differences. The emergence of Burgers and modified Korteweg-de Vries (mKdV) equations characterizes density wave dynamics in stable and unstable regions respectively. Exploration of traffic behaviors, via linear and nonlinear stability analyses, underscores the model's realism enhancement through density difference inclusion. Phase diagrams in density-sensitivity space illustrate the impact of reaction coefficients, indicating a reduction in unstable regions with increasing coefficients. Numerical simulations corroborate these findings, demonstrating heightened realism in driver behaviors near congestion as density differences are factored in. These collective results underscore the pivotal role of density discrepancies within the lattice hydrodynamic model, shaping its accuracy and reliability in traffic flow simulations.

In 2013 a research has been done upon the "Lattice hydrodynamic traffic flow model with explicit drivers physical delay" [10]. This research introduced a novel lattice hydrodynamic traffic flow model incorporating the Delay in Drivers' Sensing of Relative Flux (DDSRF) effect. Investigating the nature of traffic jamming transition through both linear stability theory and nonlinear analysis, we derive a kink-antikink soliton solution of the modified Korteweg–deVries (mKdV) equation near critical points to elucidate traffic jam dynamics. Our findings underscore the significant impact of DDSRF on traffic congestion, as evidenced by the close agreement between numerical simulations and analytical predictions. This highlights the pivotal role of drivers' delay in sensing relative flux in understanding and mitigating traffic congestion.

In 2014 a research has been done upon the "A new lattice hydrodynamic model for two-lane with the consideration of density difference effect" [11]. In this, researcher proposed a fresh lattice hydrodynamic model for two-lane traffic, integrating the Density Difference Effect (DDE). Employing both linear stability theory and nonlinear analysis, we explore the model's stability conditions and investigate jamming transitions between distinct traffic phases. Deriving the modified Korteweg–deVries (mKdV) equation near critical points, we uncover kink-antikink soliton solutions, further validated through numerical simulations. Our results demonstrate the efficacy of DDE in enhancing traffic flow stability, offering insights into potential applications across varied traffic conditions, a direction for future exploration.

In 2014, a research has been done upon a new lattice hydrodynamic model for bidirectional pedestrian flow considering the visual field effect [12]. This research introduced a novel lattice hydrodynamic model for bidirectional pedestrian flow, integrating the influence of pedestrians' visual fields. Through linear stability analysis, they established the stability conditions of the model, while employing the reductive perturbation method to derive the modified Korteweg–deVries (mKdV) equation near critical points, elucidating pedestrian jam density waves. Phase diagrams reveal transitions among various flow phases, underscoring the significant impact of visual field effects on jamming transitions. Analytical findings emphasize the pivotal role of visual information in enhancing pedestrian system stability, corroborated by numerical simulations, thus offering valuable insights into pedestrian flow dynamics.

In 2015, a research has been done upon the "Delayed-feedback control in a Lattice hydrodynamic model" [13]. In this research they delved into the application of the Delayed-Feedback Control (DFC) method in mitigating traffic congestion within a unidirectional lattice hydrodynamic traffic flow model. Employing the Hurwitz criteria and transfer function conditions, we meticulously design feedback gain and delay time parameters to stabilize traffic flow and alleviate congestion. Through Bode-plot analysis, we illustrate how the stability region expands with the implementation of delayed-feedback control, effectively curbing traffic jams. Simulation results corroborate theoretical analyses, affirming the efficacy of delayed-feedback control in stabilizing traffic flow dynamics. This study underscores the significance of considering delayed-feedback control in traffic flow modeling, with implications for future analyses on energy consumption and other pertinent aspects of traffic dynamics.

In 2015, a research has been done upon the "The control method for the lattice hydrodynamic model" [14]. This research explored the efficacy of the Delayed-Feedback Control method in addressing traffic congestion within a lattice hydrodynamic traffic flow model. Through both linear and nonlinear analyses, we derive stability conditions with and without the control signal, shedding light on the impact of feedback in stabilizing traffic flow dynamics. Numerical simulations corroborate the effectiveness of the control signal in efficiently suppressing traffic congestion, highlighting its practical utility. While acknowledging the limitations of local stability analysis in capturing the complexities of traffic flow systems, we recognize its suitability in practical applications. Future investigations may delve into achieving global stability in addressing the intricacies of traffic flow dynamics.

In 2016, a research has been done upon the "Lattice hydrodynamic model for traffic flow on curved road" [15], taking into consideration topographical features, economic considerations, and driving safety, real-world roads often exhibit curves, which significantly influence traffic flow dynamics. This study aims to elucidate the mechanisms driving traffic flow on curved roads. They propose an extended one-dimensional lattice hydrodynamic model tailored to analyze traffic flow on curved roads. Through linear stability analysis, we derive stability conditions contingent upon factors such as road curvature, friction coefficient, and entrance angle. Nonlinear density wave equations (Burgers, Korteweg–de Vries, and modified Korteweg–de Vries) characterize traffic flow behavior in stable, metastable, and unstable regions, respectively. Analytical and simulation results highlight the profound impact of road curvature, friction, and entrance angle on traffic stability, flux, and velocity. Specifically find that enlarging entrance angles and reducing curvature radius enhance traffic stability, while increasing friction exacerbates congestion. Moreover, maximal theoretical flux and velocity are influenced by these factors, suggesting nuanced considerations in road design and traffic management.

In 2016, a research has been done upon the "Lattice hydrodynamic model for two-lane traffic flow on curved road" [16]. This research analyzed the traffic dynamics on curved roads present a complex scenario, surpassing the simplicity of straight road flow. To delve into the impact of lane-changing behaviors on traffic dynamics within this context they proposed and analyze an extended lattice hydrodynamic model tailored for two-lane traffic on curved roads. Through a blend of analytical and numerical investigations, they unveil the nuanced influence of lane-changing coefficients on traffic flow stability. Deriving stability conditions via linear stability analysis highlight the role of road curvature and lane-changing behavior in shaping traffic dynamics. Characterizing nonlinear traffic behavior through the Burgers, Korteweg–de Vries (KdV), and modified KdV equations, we elucidate distinct types of jamming transitions including transitions to chaotic jams. The simulations underscore the stabilizing effect of lane-changing behaviors on traffic flow, offering insights into managing traffic dynamics on curved roads to mitigate chaotic phenomena effectively.

In 2016, a research has been done upon the "Lattice hydrodynamic model based traffic control: A transportation cyber–physical system approach" [17]. This research analyzed a lattice hydrodynamic model stands as a cornerstone in understanding traffic flow dynamics, particularly in delineating jamming transitions. Prior investigations have highlighted the potential of control methods in enhancing traffic conditions within this framework. In this study, we introduce a novel control approach within the lattice hydrodynamic model, adopting a transportation cyber–physical system perspective, wherein control is implemented at a single lattice site. Through comprehensive simulations, they validated the efficacy of this approach, ensuring the smooth and efficient operation of traffic flow. Our work not only presents a robust traffic congestion control scheme but also offers insights into advancing traffic flow operations within transportation cyber–physical systems. Future endeavors should address practical complexities such as multi-lane scenarios, time delays, and measurement errors in refining control loop designs.

In 2017, a research has been done upon the "A lattice hydrodynamic model based on delayed feedback control considering the effect of flow rate difference" [18]. This research introduced a lattice hydrodynamic model that not only incorporates the impact of flow rate differences but also integrates delayed feedback control signals, providing a more comprehensive understanding of traffic dynamics. By employing control methods analyze the stability of the model and deduce critical conditions for linear steady traffic flow. Through numerical simulations demonstrate the advantages of our proposed model with and without flow rate difference effects and control signals, aligning closely with theoretical analyses. The findings underscore the efficacy of the proposed delayed feedback control in mitigating traffic congestion, offering insights into its broader applicability in addressing challenges across the global transportation network, including energy consumption issues.

In 2017, a research has been done upon the "New control strategy for the lattice hydrodynamic model of traffic flow" [19]. This research introduced a novel delayed-feedback control strategy, this study applies it to a lattice hydrodynamic model of traffic flow, incorporating the variation rate of optimal velocity as the control signal. Through frequency-domain analysis with control theory derive linear stability conditions. Subsequent simulations in both periodic boundary and on-ramp scenarios demonstrate the efficacy of this control strategy in mitigating traffic congestion. Theoretical analyses and simulations jointly affirm the positive impact of the control signal in suppressing traffic jams, underscoring its potential in enhancing traffic management strategies.

In 2017, a research has been done upon the "congested traffic patterns of two-lane lattice hydrodynamic model with on-ramp" [20]. This research endeavored to replicate real-world congested traffic scenarios, particularly those induced by on-ramps, through a microscopic traffic model. This research enhanced two-lane lattice hydrodynamic traffic flow model, designed to rectify issues such as vehicles moving backward—a common limitation in existing models. By incorporating deterministic and stochastic on-ramps into our framework, they successfully reproduce a spectrum of observed empirical congested patterns, including moving localized clusters, stop-and-go traffic, pinned localized clusters, oscillating congested traffic, and homogeneous congested traffic.

These findings demonstrate the model's efficacy in predicting various congested traffic patterns. While our study provides a foundational methodology for traffic management, future research should focus on validating the model against empirical data, leveraging resources like NGSIM trajectory data for calibration. Additionally, extensions to the model could involve incorporating mixed manual and automated traffic scenarios and accounting for more complex geometries, opening avenues for further exploration and refinement in traffic modeling and management.

In 2018, a research has been done upon the "Feedback control method in lattice hydrodynamic model under honk environment [21].This research implemented a feedback control mechanism influenced by traffic flux within a lattice hydrodynamic model of traffic flow under honk-induced conditions, this study investigates its impact on traffic flux and stability. Leveraging control theory, they derive linear stability conditions, complemented by numerical simulations to assess the honk effect on traffic flow dynamics. Our findings indicate that feedback gains influenced by honk-induced effects effectively stabilize traffic flow and mitigate congestion. This highlights the importance of considering honk-induced effects, particularly traffic flux, within lattice hydrodynamic models. Future research will delve into identifying appropriate control parameters and honk effect ranges, enriching our understanding of traffic dynamics under honk environments.

In 2018, a research has been done upon the "An improved lattice hydrodynamic model considering the backward looking effect and the traffic interruption probability" [22]. This research introduced an enhanced lattice hydrodynamic model incorporating the "backward looking" effect and traffic interruption probability, enhancing traffic flow stability. Linear stability analysis yields stability criteria, while nonlinear theory produces the mKdV equation to describe traffic congestion. Numerical simulations confirm the efficacy of these factors in improving traffic flow stability, demonstrating faster density reduction compared to previous models. This underscores the significance of considering traffic interruption probability and the "backward looking" effect in optimizing traffic flow stability.

In 2018, a research has been done upon the "Analyses of lattice hydrodynamic model using delayed feedback control with passing" [23]. This research employed a delayed feedback control method in a one-dimensional lattice model, integrating the effect of passing, to assess system stability. Utilizing control theory, we derive stability conditions using Hurwitz criteria and obtain the norm of transfer functions. Through Bode plot analysis, they demonstrate the efficacy of the control signal in minimizing jamming transitions caused by passing, ensuring flow stability even at higher passing rates. Our simulations validate these theoretical findings, indicating a significant reduction in congestion with the application of the control parameter. Additionally, nonlinear analysis yields the mKdV equation, with conditions identified for cases where its derivation is not feasible, particularly at higher passing rates. While higher passing rates tend to destabilize the system, the control parameter effectively mitigates chaos, ensuring system stability. This underscores the validity of our approach, highlighting the potential of delayed feedback control in managing congestion during passing scenarios. Future extensions of our model could explore additional phenomena such as anticipation effects and interruption probabilities, particularly in the context of curved roads.

In 2019, a research has been done upon the "A new lattice hydrodynamic model accounting for the traffic interruption probability on a gradient highway [24]. This research introduced a novel lattice hydrodynamic model that incorporates traffic interruption probability on a gradient highway. Linear analysis reveals that both traffic interruption probability and slope significantly influence the stability region. Nonlinear analysis yields the mKdV equation, capturing phase transitions in traffic flow. Numerical simulations confirm the analytical findings, demonstrating the effective improvement of traffic flow dynamics by considering traffic interruption probability on gradient highways.

In 2019, a research has been done upon the "An extended lattice hydrodynamic model considering the delayed feedback control on a curved road" [25]. This research explored the impact of delayed feedback control on curved roads, both analytically and numerically introduced an extended lattice hydrodynamic model for single-lane roads, incorporating comprehensive information. Utilizing control methods, we establish the stability of the model and deduce critical conditions for linear steady traffic flow. Numerical simulations validate the efficacy of our proposed model with and without delayed feedback control, comparing results on curved and straight roads. The findings underscore the feasibility of the control signal in mitigating traffic jams, highlighting its potential for effective traffic management.

In 2019, a research has been done upon the "The impact of the individual difference on traffic flow under honk environment in lattice hydrodynamic model" [26]. This research investigated the impact of individual differences in drivers' characteristics under honk-induced environments on single-lane traffic flow, employing a feedback control method. Leveraging modern control techniques, they conducted linear stability analysis incorporating feedback terms. Numerical simulations elucidate the diverse roles played by individual driver characteristics in traffic flow stability under honk conditions. Their findings, supported by the Hurwitz stability criterion, underscore the beneficial influence of skillful driving behaviors in enhancing traffic flow dynamics amidst honk-induced environments.

In 2020, a research has been done upon the "A novel lattice hydrodynamic model accounting for driver's memory effect and the difference of optimal velocity on curved road [27]. In this research, novel lattice hydrodynamic model consider the influence of driver memory and variations in optimal velocity on curved roads was introduced. Through linear stability analysis, they derive stability conditions for the model. Employing nonlinear analysis, they deduce the mKdV equation to examine jam evolution near the vertex. Additionally, they derive the exact solution of the mKdV equation. Numerical examples further elucidate the impact of memory effect and optimal velocity differences on traffic stability. Both analytical and numerical findings converge, demonstrating the efficacy of these factors in alleviating traffic jams on curved roads.

In 2020, a research has been done upon the "A feedback control method with consideration of the next-nearest-neighbor interactions in a lattice hydrodynamic model" [28]. This research investigated a feedback control method considering next-nearest-neighbor interactions in a lattice hydrodynamic traffic model. Through linear stability analysis, stability conditions are derived using first-order and second-order transfer functions. Nonlinear analysis yields the mKdV equation describing density wave behavior. Theoretical analysis underscores the impact of feedback gain, nearest-neighbor weight, and next-nearest-neighbor weight on traffic flow stability. Numerical simulations confirm these findings, showing the effectiveness of the control strategy in mitigating traffic jams, with enhanced stability observed in the extended model compared to existing ones. Analyzing scatter plots of traffic density difference reveals chaotic patterns, mitigated by the consideration of neighboring interactions. Traffic hysteresis loops demonstrate improved stability, indicating efficient traffic control. The study concludes that attention to neighboring interactions aids in stabilizing traffic flow, suppressing congestion effectively.

In 2020, a research has been done upon the "A new lattice hydrodynamic model for bidirectional pedestrian flow with consideration of pedestrians' honk effect [29]. This research understands a pedestrian behavior crucial for effective traffic simulation and facility design. In this context, interactions among pedestrians can mimic a virtual honk effect, encouraging others to move faster in crowded environments. To capture this phenomenon accurately, we propose a novel lattice hydrodynamic model for bidirectional pedestrian flow, incorporating pedestrians' honk effect. Introduce the concept of critical density to define honk occurrences and derive the stability condition of the model through linear stability analysis, revealing the significant impact of honk effects on pedestrian flow stability. Nonlinear stability analysis yields the modified Korteweg–de Vries (mKdV) equation, from which we obtain the kink-antikink soliton wave, elucidating congestion propagation near the neutral stability curve. Simulation results demonstrate that pedestrians' honk effect efficiently alleviates crowding and enhances the stability of bidirectional pedestrian flow.

In 2020 a research has been done upon the "Lattice hydrodynamic modeling with continuous self-delayed traffic flux integral and vehicle overtaking effect" [30]. In this research introduced a novel lattice hydrodynamic model incorporating vehicle overtaking and continuous self-delayed traffic flux integration. Linear stability analysis yields the stability condition, indicating that increasing the delay time step enlarges the stable region. Nonlinear analysis results in the modified Korteweg–de Vries (mKdV) equation, describing traffic density wave propagation near critical points, with kink–anti-kink solutions derived for different passing constants. Results reveal two cases: under a threshold passing constant (Case I), uniform flow and kink jam phases occur, while exceeding the threshold (Case II) leads to jamming transitions from uniform flow to chaotic traffic waves. Simulation examples validate these findings, showing reduced density fluctuations with increased delay time and chaotic traffic flow under specific conditions.

In 2021, a research has been done upon the "New feedback control for a novel two-dimensional lattice hydrodynamic model considering driver's memory effect [31]. This study presented a novel two-dimensional lattice hydrodynamic model integrating driver memory effects and a new feedback control signal. Linear stability analysis using control methods yields stability conditions, while nonlinear analysis provides the kink–antikink solution of the modified Korteweg–de Vries (mKdV) equation for density wave description. Numerical simulations validate theoretical findings, demonstrating the efficacy of the new control signal in stabilizing traffic flow. Conversely, increased driver memory time leads to greater traffic flow instability. Overall, the proposed model offers insights for mitigating traffic congestion and enhancing road capacity.

In 2021, a research has been done upon the "Effect of self-stabilizing control in lattice hydrodynamic model with on-ramp and off-ramp [32]. This research introduced a novel hydrodynamic model considering dual ramps—a combination of an on-ramp and an off-ramp—and investigates their positional relationship's influence on traffic flow stability. Building on prior research indicating the positive impact of historical speed and headway data on single-lane traffic, historical flow information is incorporated to address road instability due to ramp crossings. Linear analysis theory yields the model's neutral stability curve and critical condition, validated through numerical simulations. Key findings include the negative impact of both incoming and outgoing flows on the main road, the stabilizing effect of minimizing the distance between consecutive ramps, and the efficacy of a self-stabilizing control strategy in improving road stability amidst ramp constructions.

In 2021, a research has been done upon the "Delay-independent traffic flux control for a discrete-time lattice hydrodynamic model with time-delay" [33]. This research introduced a discrete-time lattice hydrodynamic model featuring time-delay and a control signal for traffic flux. The model's stability is analyzed using the discrete-time Lyapunov stability theorem, yielding a sufficient condition presented in a linear matrix inequality format. Through numerical simulations, various scenarios of time-delay and control gain are explored, revealing that time-delay exacerbates traffic congestion while the discrete-time traffic flux control signal enhances traffic flow stability.

In 2021, a research has been done upon the "A new feedback control scheme for the lattice hydrodynamic model with [34]. This research amidst growing concerns over traffic congestion's economic and environmental impacts, this paper presents a novel hydrodynamic traffic flow model integrating drivers' sensory memory and a new feedback control scheme. Stability analysis reveals how drivers' memory influences instability regions, with the proposed control scheme effectively reducing instability. Numerical simulations further validate these findings, shedding light on short- and long-term traffic behaviors and hysteresis loops.

In 2022, a research has been done upon the "Energy consumption in a new lattice hydrodynamic model based on the delayed effect of collaborative information transmission under V2X environment" [35]. This research introduced a novel lattice hydrodynamic model incorporating collaborative delayed information transmission of density and flux, offering insights into traffic flow dynamics. Theoretical analysis, including linear and nonlinear methods, reveals that this collaborative effect effectively mitigates congestion, reduces energy consumption, and enhances traffic system stability. Numerical simulations corroborate these findings, demonstrating the model's efficacy in improving traffic flow efficiency and reducing energy consumption. Overall, the study underscores the potential of collaborative delayed information transmission in addressing traffic congestion and improving traffic system performance.

In 2022, a research has been done upon the "Robust H-infinity control for connected vehicles in lattice hydrodynamic model at highway tunnel" [36]. This research introduced a novel macro hydrodynamic model tailored for tunnel traffic within a networked environment. Utilizing characteristics specific to tunnel traffic, a lattice hydrodynamics model is established alongside a corresponding control strategy informed by connected vehicle data. Theoretical analyses yield internal stability conditions for the traffic system, while considering external disturbances from the tunnel prompts the proposal of a robust H-infinity control strategy and associated stability conditions. String stability analysis ensures non-amplification of instantaneous disturbances. Numerical simulations comparing controlled and uncontrolled scenarios demonstrate the efficacy of the proposed control strategy in maintaining tunnel traffic system stability, mitigating congestion, and preventing disturbance amplification.

In 2022, a research has been done upon the "Study on Energy Dissipation and Fuel Consumption in Lattice Hydrodynamic Model under Traffic Control" [37]. This research proposed an extended one-dimensional lattice hydrodynamic model incorporating feedback control based on the average optimal flow of multiple grids downstream. Through linear stability analysis, the model's stability condition is derived. Nonlinear analysis yields the mKdV equation and its kink-antikink density wave solution. Numerical simulations explore density wave variations, energy dissipation, and fuel consumption under traffic control. Findings indicate that feedback control effectively mitigates congestion, enhances traffic stability, and reduces energy dissipation and fuel consumption. However, limitations arise due to the inability to observe traffic conditions far downstream. Integration with intelligent transportation systems can address this limitation, facilitating traffic forecasting and management.

In 2022, a research has been done upon the "Analyses of lattice hydrodynamic area occupancy model for heterogeneous disorder traffic" [38]. This research talked about developing countries traffic diverse and disorderly, comprising various vehicle types such as automobiles, trucks, buses, and motorbikes. Managing these complex transport networks relies heavily on modeling mixed (heterogeneous) traffic dynamics. To address this, a new lattice model is proposed, considering the area occupancy of different vehicles. Stability analysis reveals the model's efficacy, with phase diagrams linking traffic stability to vehicle fractions. Using reduction perturbation, the behavior of disordered traffic is explored, leading to the derivation of the mKdV equation near critical points. Numerical simulations confirm that a higher fraction of small vehicles stabilizes traffic flow. While this study focuses on one-lane traffic without overtaking effects, future research could extend to multi-lane and two-dimensional traffic dynamics to better understand heterogeneous traffic efficiency.

In 2023, a research has been done upon the "The jamming transition of multi-lane lattice hydrodynamic model with passing effect" [39]. This is in context of multi-lane highways where vehicles can freely switch lanes and overtake each other. They introduce a modified multi-lane lattice hydrodynamic model that considers the passing effect. Using the reduction perturbation method, we derive the stability norm of the model, revealing a positive correlation between the total number of lanes and traffic flow stability. Nonlinear stability analysis yields the modified Korteweg-de Vries (mKdV) equation, highlighting the formation and transmission process of traffic jams near the neutral stability curve. Numerical simulations validate the theoretical findings. The study addresses the passing behavior in multi-lane environments, shedding light on traffic flow dynamics. However, it assumes homogeneous traffic flow and static parameters, which may not fully reflect real-world scenarios. Future research aims to address these limitations.

In 2023, a research has been done upon the "Heterogeneous lattice hydrodynamic model and jamming transition mixed with connected vehicles and human-driven vehicles" [40]. This research is with the rapid advancement of connected vehicle (CV) technologies alongside human-driven vehicles (HDVs) on roadways, a need arises to understand the dynamics of mixed vehicular flow. To address this, researchers introduced a heterogeneous lattice hydrodynamics model that incorporates differences in information acquisition between CVs and HDVs. Through linear stability analysis, we establish stability criteria for the model. Nonlinear stability analysis leads to the derivation of a modified Korteweg–de Vries equation (mKdV), offering insights into traffic jam formation. Numerical simulations examine the impact of CV penetration rates and directional visual fields on traffic flow stability, confirming theoretical conclusions. However, the study overlooks unconventional driving behaviors and neglects higher-order nonlinear terms in theory derivation. Moreover, the model lacks validation against real-world traffic data. Future research will focus on addressing these limitations to enhance model accuracy and applicability.

In 2023, a research has been done upon the "An Extended Multilane Lattice Hydrodynamic Model Considering the Predictive Effect of Drivers under Connected Vehicle Environment [41]. This research introduced a modern highway system multilane roads are a common sight offering drivers increased flexibility in lane choice and maneuverability. The advent of connected vehicle technology enables drivers to anticipate traffic conditions and adjust their driving behavior accordingly. This study delves into the predictive capabilities of drivers in multilane scenarios using a lattice hydrodynamic model. Through reductive perturbation analysis, derive stability criteria for the model, and in cases of instability, the modified Korteweg–de Vries (mKdV) equation offers insights. Solving this equation yields the kink–antikink soliton wave solution, shedding light on traffic jam dynamics. Findings underscore the significant impact of lane count and driver prediction time on traffic flow stability. Simulation results reveal that as the number of lanes increases or prediction time lengthens, traffic density fluctuations decrease, eventually leading to uniform flow conditions. This study highlights the critical role of driver anticipation in maintaining stable traffic flow on multilane highways.

In 2023, a research has been done upon the “The dynamic evolution integrating the flux limit effect in lattice hydrodynamic model on two lanes under V2X environment [42]. This research introduced traffic volumes rise, implementing road flux limits is crucial for traffic control. This study presents a lattice hydrodynamic model integrating the flux limit effect (FLE) in a two-lane system with V2X communication. Through linear stability analysis, stability conditions related to FLE are established, showing its effectiveness in widening stability margins during lane changes. Numerical simulations demonstrate FLE's impact on density waves, fuel consumption, and emissions, highlighting its role in enhancing traffic stability, particularly during lane changes. Managing traffic flow becomes increasingly critical with rising volumes, making road flux limits pivotal. The study introduces a novel lattice model to address FLE in two-lane systems, emphasizing its benefits in enhancing traffic flow stability. Refining parameter calibration methods will be essential for optimal FLE implementation in such systems going forward.

In 2024, a research has been done upon the “Impact of the visibility effect on phase transitions in lattice hydrodynamic model under the bad weather traffic environment” [43]. This research is focused on adverse weather condition as reduced visibility can significantly impact traffic flow dynamics. To explore this effect, we develop a new lattice hydrodynamic model tailored to assess visibility's influence on traffic. Through density-sensitivity phase analysis, they observe that as visibility decreases, the stable region of traffic flow expands. Nonlinear analysis yields the kink-antikink soliton solution of the mKdV equation, directly linked to visibility's effect. Numerical simulations further validate these findings, demonstrating improved traffic stability amid reduced visibility, albeit at the cost of decreased traffic flux. Proposed lattice model integrates the visibility effect on traffic flux, investigated via both linear and nonlinear analyses. Linear stability analysis provides stability conditions and neutral stability curves, while nonlinear analysis yields the mKdV equation, characterizing traffic density wave propagation. Numerical simulations corroborate theoretical insights, highlighting enhanced traffic stability under reduced visibility. Intriguingly, while traffic flow stability improves, traffic flux decreases to some extent as visibility decreases. This mirrors real-world scenarios, where reduced traffic flow ensures driving safety, affirming our model's fidelity to reality. In 2024 “Phase transition of traffic congestion in lattice hydrodynamic model: Modeling, calibration and validation” [44]. This research introduced a real-world transportation system, the time it takes for drivers to react to stimuli from the vehicle ahead varies depending on their speed. Traditional models for following cars often treat this reaction time as a fixed constant, which doesn't account for the diverse nature of drivers' responses. To address this limitation, this study proposed a sigmoid function to represent how drivers' response times change with their current speed. By integrating this nonlinear function into a new hydrodynamic lattice model, they created a more realistic representation of traffic flow. The model's properties are explored using Fourier series analysis, and they derived a criterion for its linear stability. To validate its effectiveness, numerical simulations are conducted on a circular road. Results demonstrate that the model accurately reproduces the formation and propagation of density waves observed in real traffic scenarios. Furthermore, we calibrate and validate the model using empirical data, finding strong quantitative agreement between simulation outcomes and detector data. This research enhances our understanding of traffic dynamics and offers a valuable tool for predicting and managing real-world traffic patterns.

Conclusion

This literature review underscores significant research gaps in the investigation of lattice hydrodynamic models, highlighting the need for comprehensive approaches to urban traffic and pedestrian flow modeling. Many existing models, particularly those emphasizing the "backward-looking" effect, focus on a single following vehicle to enhance traffic stability neglecting the complexities of multi-vehicle interactions. Urban factors, such as intersections, multi-lane traffic and varying driver behaviors remain inadequately addressed limiting the models' applicability in real-world scenarios. Similarly, bidirectional pedestrian flow models oversimplify interactions failing to account for individual behaviors varying walking speeds and environmental obstacles—crucial elements for accurately representing urban dynamics. The literature emphasizes the necessity for models that capture the nuances of human behavior across diverse conditions to support effective urban planning. Furthermore, existing models often overlook mixed traffic types and diverse road conditions focusing instead on vehicle adjustments to improve efficiency. The integration of connected and automated vehicles as well as the complexities of driver behaviors warrants further exploration. While non-lane-based lattice models address lateral effects they overlook critical interactions in multi-lane traffic and real-world conditions like intersections. Additionally, while integrating driver forecast effects (DFE) into lattice models enhances stability it does not sufficiently account for the complexities of multi-lane traffic. Advanced traffic management systems must be considered in future research to bridge these gaps. Moreover, understanding individual driver differences such as honking behavior can refine traffic management strategies. Overall, although current studies provide valuable insights substantial gaps persist in addressing the complexities of urban traffic and pedestrian dynamics. Future research should prioritize diverse traffic conditions adaptive control mechanisms and empirical validation to create effective models that inform planning ultimately reducing congestion, pollution, and travel times in cities.

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